
N. Sekarapandian, Y. V. S. S. Sanyasiraju, S. Vengadesan

Department of Applied Mechanics, Indian Institute of Technology, Madras, Chennai, India
Department of Mathematics, Indian Institute of Technology, Madras, Chennai, India

Online publication date: 07 December 2009


To link to this Article: DOI: 10.1080/10407780903423932
URL: http://dx.doi.org/10.1080/10407780903423932

Full terms and conditions of use: http://www.informaworld.com/terms-and-conditions-of-access.pdf

This article may be used for research, teaching and private study purposes. Any substantial or systematic reproduction, re-distribution, re-selling, loan or sub-licensing, systematic supply or distribution in any form to anyone is expressly forbidden.

The publisher does not give any warranty express or implied or make any representation that the contents will be complete or accurate or up to date. The accuracy of any instructions, formulae and drug doses should be independently verified with primary sources. The publisher shall not be liable for any loss, actions, claims, proceedings, demand or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of this material.
A NOVEL SEMI-EXPLICIT SPATIALLY FOURTH ORDER ACCURATE PROJECTION METHOD FOR UNSTEADY INCOMPRESSIBLE VISCOUS FLOWS

N. Sekarapandian1, Y. V. S. S. Sanyasiraju2, and S. Vengadesan1
1Department of Applied Mechanics, Indian Institute of Technology, Madras, Chennai, India
2Department of Mathematics, Indian Institute of Technology, Madras, Chennai, India

This article describes a simple and elegant compact higher order finite-difference based numerical solution technique to the primitive variable formulation of unsteady incompressible Navier Stokes equations (UINSE) on staggered grids. The method exploits the advantages of the D’yakanov ADI-like scheme and a non-iterative pressure correction based fractional step method. Spatial derivatives are discretized to fourth order accuracy and the time integration is realized through the Euler explicit method. The fast and efficient iterative solution to the discretized momentum and pressure Poisson equations is achieved using a variant of conjugate gradient method. Spatial accuracy and robustness of the solver are tested through its application to relevant benchmark problems.

1. INTRODUCTION

The higher order compact schemes (HOCS) suitable to the discretization of convection diffusion equations (CDE) can be derived using polynomial or exponential series expansions. In the context of Navier Stokes equations (NSE), the vorticity transport equation (VTE) pertaining to the vorticity-stream function (VS) formulation can be exactly classified as a CDE; whereas, the analogous momentum equations in primitive variable (PV) formulation only resembles the CDE because of the presence of pressure gradient term. It is due to this obvious reason, that developed HOC polynomial [1–10] and exponential schemes [11] are straightforward in their implementation to the VS formulation. However, the requirement of three-dimensional simulations makes the consideration of extending these schemes to the primitive variable form of UINSE inevitable. A fully implicit ADI algorithm applying higher order Padé schemes to momentum equations in CDE sense has been reported in reference [12]. This approach is feasible through the usage of the fractional step based projection method which was first proposed in reference [13]; wherein, the pressure gradient term is eliminated while solving momentum equations to compute pseudovelocity field.
Generally, the higher order Padé schemes when applied over a set of discrete points are conceived in matrix form and can be broadly classified as central difference-based compact schemes (CDCS) and upwind difference-based compact schemes (UDCS), as proposed in references [14–16] and [17–19], respectively. Usually, the particular choice of the compact scheme is determined by the physics of the problem to be simulated. The central difference (CD)-based compact schemes are non-dissipative in nature but may cause spurious oscillations in the convection dominated flows. This drawback can be subdued by inducing artificial viscosity into these schemes [16]. But the aliasing error that may occur due to the excess in numerical viscosity has to be controlled using compact filtering schemes, and so, CD-based compact schemes are suitable for LES applications [20, 21]. In the case of numerical simulation of wave propagation or acoustic problems, the higher order upwind schemes are indispensable [18, 19, 22].

In this work, we have adopted CD compact schemes keeping the focus on LES applications in the future. Similarly, an essential comparison on the performance of compact schemes in the staggered and collocated grids, made in reference [20], shows that the former has better conservation properties of mass and momentum. In addition, it is pointed out in references [23, 24], when the skew symmetric form of
the convection term is used on a staggered grid, the unsteady incompressible flow simulation will be stable and free of numerical dissipation, due to the satisfaction of conservation of kinetic energy. These advantages substantiate the choice of staggered grid and the skew symmetric form of the convection term in the computations.

The next vital issue to be discussed is the time step restriction while using the compact schemes. Previous works [23, 25], signify that the higher order spatial accuracy pays penalty in terms of shrinking the CFL limit. Keeping note of this fact, we have adopted a semi-explicit form of discretization to the momentum equations. This selection allows a larger CFL limit than that for a fully explicit form, and increases the sparsity of the iterative matrix which is denser in the case of fully implicit discretization.

In this work we have introduced a new approach of using higher order Padé schemes other than the fully implicit scheme given by reference [12] to solve the UINSE. In this direction, some of the recent works [24, 26–28] show the commonality in using fractional step or a pressure correction-based method to solve the UINSE, each of which differs in the procedure to satisfy the incompressibility constraint.

It is well known that in the projection method approach, in order to satisfy the incompressibility constraint, it is imperative to solve a Poisson equation for pressure or pressure correction. In the literature, several methods have been reported to solve for the pressure or pressure correction. One of the approaches is to convert the Poisson equation, which is of an elliptic kind, to a parabolic problem [26] using pseudo transient. In another approach [27], the local cell by cell pressure correction and global pressure correction strategies are followed. The comparison shows that the former performs better for low Reynolds number, but the latter is the attractive alternative for high Reynolds number flows. To reduce the computational cost, the matrix diagonalization technique has been recommended in reference [28] to solve the Poisson Neumann problem. In the work of reference [24], it is pointed out that using a compact operator to the Laplacian either for the second derivative or in the form of divergence of a gradient may lead to the singularity of the discrete matrix. But the fourth order convergence can be compensated by using a conventional second order operator and performing inner iterations to the point where the source term of the Poisson equation vanishes towards zero [29]. On the collocated grids, the Laplacian is discretized first using a compact second derivative approximation on a finer grid, and then with two time application of first derivative approximation on coarser grids. The reason for such an implementation is explained in reference [30].

The purpose of this work is to test the utility of a semi-explicit fractional step method, suggested by reference [31], when combined with advantages of matrix form of compact finite-difference schemes. There are two main unique implementations in this article. First, instead of using conventional Peaceman-Rachford type ADI schemes, the multidimensional PDE is solved using the locally one-dimensional approach suggested by D’yakanov [32]. The main advantage of this factorization is that the tridiagonal nature of the coefficient matrix for the derivatives can be retained at all times so that the matrix inversion is cost effective. The second unique feature is a new approach of deriving a discrete equation for pressure correction using the continuity equation. Moreover, owing to the elliptic nature of the discrete
momentum and Poisson equations, the efficient iterative procedure of BICGSTAB
(2), discussed in reference [33], has become viable to obtain their solutions. The
organization of the rest of this article is as follows. The governing equations and
the numerical scheme are explained in section 2. Section 3 elucidates the implemen-
tation of the D’yakanov factorization procedure into the discrete momentum
equations, and discusses the details of the compact finite-difference (FD) approxima-
tions (along with the appropriate staggered grid schematics), used to discretize the
spatial derivatives contained in these equations. The distinct derivations of
the discrete pressure Poisson equation of second and fourth order accuracies
and their implementation is discussed in section 4. In section 5, the performance
of the developed numerical scheme is validated against a few benchmark problems.
The conclusions are drawn in the last section.

2. OUTLINE OF THE NUMERICAL SCHEME

2.1. Governing Equations

Generally, the incompressible Navier-Stokes equations denote the continuity
and the momentum equations. These equations in vector form in Cartesian coordi-
nates are given by

\[
\begin{align}
U_t + \nabla p &= -(U \cdot \nabla)U + \frac{\nabla^2 U}{\text{Re}} \quad (1) \\
\nabla \cdot U &= 0 \quad (2)
\end{align}
\]
on \Omega \subset \mathbb{R}^d \times \mathbb{R}^+, where \(d\) is the dimension of the problem, \(U\) is the velocity vector, \(p\) is
the pressure, \(\text{Re}\) is the Reynolds number defined by \(\text{Re} = \frac{Ulc}{\nu}\), \(U_c\) is the characteristic
velocity, \(lc\) is the characteristic length, and \(\nu\) is the kinematic viscosity. Solving
Eqs. (1) and (2) require some initial and boundary conditions. Such general
conditions are represented by \(U(X, 0) = U_0\) and \(U|_{\partial \Omega} = U_b\), respectively.

2.2. Discrete Formulation of Governing Equations

The second order, time-discrete semi-explicit form of Eqs. (1) and (2) can be
written as

\[
\begin{align}
\frac{U_{n+1} - U^n}{\Delta t} + \nabla p^{n+\frac{1}{2}} &= -((U \cdot \nabla)U)^{n+\frac{1}{2}} + \frac{1}{\text{Re}} \nabla^2 (U^{n+1} + U^n) \quad (3) \\
\nabla \cdot U^{n+1} &= 0 \quad (4)
\end{align}
\]
subjected to the boundary condition \(U^{n+1}|_{\partial \Omega} = U^{n+1}_b\). The spatially discretized
version of the coupled system Eqs. (3) and (4) is cumbersome to solve directly. Therefore,
a fractional step procedure is used to approximate the solution of the coupled
system by first solving an analog to Eq. (3), (excluding the divergence constraint) for
an intermediate quantity $U^n$. Then $U^n$ is projected onto the space of divergence free field to get $U^{n+1}$. The sequence of steps of the algorithm is as follows.

**Step 1:** Computation of $U^n$

\[
\frac{U^* - U^n}{\Delta t} + \nabla q = - (\nabla \cdot U) U^{n+\frac{1}{2}} + \frac{1}{2\text{Re}} \nabla^2 (U^* + U^n) \tag{5}
\]

and

\[
U^*|_{\partial\Omega} = U_b^{n+1} \text{ if } \nabla q = \nabla p^{n-\frac{1}{2}} \tag{6}
\]

\[
(\nabla \cdot U) U^{n+\frac{1}{2}} = \frac{3}{2} (\nabla \cdot U) U^n - \frac{1}{2} (\nabla \cdot U) U^{n-1} \tag{7}
\]

In Eq. (5), the convection and diffusion terms are discretized using Adams-Bashforth and Crank-Nicholson schemes, respectively.

**Step 2:** Projection

\[
U^* = U^{n+1} + \Delta t \nabla \phi^{n+1} \tag{8}
\]

where $\phi$ is the pressure correction, computed by enforcing $U^{n+1} = 0$.

**Step 3:** Updating of the pressure

\[
p^{n+\frac{1}{2}} = q + K \phi^{n+1} \tag{9}
\]

where $K = (I - \frac{\Delta t}{\text{Re}} \nabla^2)$ if $q = p^{n-\frac{1}{2}}$.

### 3. IMPLEMENTATION

#### 3.1. Compact Schemes in Matrix Form

In this section, our approach to develop a compact scheme in matrix form is briefly described. In general, in the compact schemes a linear combination of derivatives is equated with a linear combination of function values. In the case of a structured Cartesian grid, if a line by line marching approach is considered, then on every line enclosing the boundary nodes there will be several interior nodes. Therefore, the approximation of the derivatives using compact schemes, on all nodes along a particular line, involves an implicit solution procedure and is represented in the matrix equation as,

\[
[A]F = [B]f \tag{10}
\]

where $[A]$ and $[B]$ are the coefficient matrices and $F$ and $f$ are the column vectors of the derivative and function values, respectively. The decision of either including or excluding the boundary nodes in the system Eq. (10) has to be made depending on the requirement of imposing Dirichlet, Neumann, periodic, or nonperiodic boundary conditions. In the next subsection, we demonstrate the utilization of the
compact FD schemes, in the form of Eq. (10), to factorize the discrete momentum equations given by Eq. (5).

### 3.2. D’yakanov ADI Procedure for Discrete Momentum Equation

In Eq. (5), the convection and diffusion terms have been subjected to an explicit and implicit treatment, respectively, and so its solution involves a matrix inversion. In particular, the coefficient matrix of the unknown vector in Eq. (5) is of the form \((I - \Delta)\), where \(I\) is the identity matrix and \(\Delta\) is the Laplacian. For example, the second order discretization of this matrix yields a banded pentadiagonal matrix which should be iteratively inverted using methods like the strongly implicit procedure (SIP). Furthermore, the compact fourth order discretization yields a full matrix, whose inversion is computationally very expensive. Therefore, in this work we have factorized the implicit part in a similar fashion to that of reference [12], which is explained below. Considering for example, the \(u\) momentum equation from Eq. (5) and factorizing it in two dimensions yields,

\[
\left( 1 - \frac{\Delta t}{2Re} \frac{\partial^2}{\partial x^2} \right) \left( 1 - \frac{\Delta t}{2Re} \frac{\partial^2}{\partial y^2} \right) u_{i,j}^* = - \frac{\partial p_{n-1}^*}{\partial x} - \frac{3}{2} (\nabla \cdot U) u^n + \frac{1}{2} (\nabla \cdot U) u^{n-1} + \frac{1}{2Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^n
\]

The calculation of the pseudo velocity \(u^*\) from Eq. (11) involves the matrix manipulations in its LHS. Herein, we apply the D’yakanov ADI-like procedure making use of Eq. (10). Now, the compact approximations of the derivatives in the RHS of Eq. (11) are estimated a priori due to their explicit treatment and stored in say \(c_{i,j}\). Then, Eq. (11) can be written as

\[
A^{-1}_{xx} T_{xx} A^{-1}_{yy} T_{yy} u_{i,j}^* = c_{i,j}
\]

where \(T_{xx} = (A_{xx} - \frac{\Delta t}{2Re} B_{xx})\) and \(T_{yy} = (A_{yy} - \frac{\Delta t}{2Re} B_{yy})\). The symbols \(A_{xx}, A_{yy}\) represent the coefficient matrices of the second derivatives and \(B_{xx}, B_{yy}\) represent the coefficient matrices of the function values in \(x\) and \(y\) directions, respectively.

Eq. (12) is solved in two stages which are analogous to LU decomposition and is given by

\[
A^{-1}_{yy} T_{yy} u_{i,j}^* = z_{i,j}
\]

\[
A^{-1}_{xx} T_{xx} z_{i,j} = c_{i,j}
\]

The first stage involves two steps. In the first step, the quantity \(z_{i,j}\) is calculated using Eq. (13) and Eq. (6) for the boundary values of \(u^*\) along \(y\) direction at the physical \(x\) boundaries of the computational domain. The calculations in this step only involve the tridiagonal matrix inversion, which is computationally cost effective. Subsequently, in the second step Eq. (14) is solved along the constant \(y\) lines, to compute
for all interior nodes, making use of $z_{i,j}$ already computed along boundary nodes in the first step. In the second stage, the pseudo velocities are calculated for all interior nodes in the computational domain by solving Eq. (13) along the constant $x$ lines. Here it is worth mentioning that in the second step of the first stage and in the second stage the bandwidth of the coefficient matrix may slightly increase from that of a tridiagonal matrix. Therefore, in these calculations the BICGSTAB (2) algorithm has been used to obtain a faster converged solution.

3.3. Details of the Compact System for Various Spatial Derivatives

The schematic of the staggered Cartesian grid used in the simulations is given in Figure 1. The horizontal and vertical stencil arrangements to deduce the compact systems of form Eq. (10) for all $x$ and $y$ partial derivatives of Eq. (11), have to be identified from this figure. In the subsequent discussion of this subsection, the details of the compact system used to discretize the convection, diffusion, and the pressure gradient terms of Eq. (11) are elaborated.

3.3.1. Convection term. Convection terms can be represented in four different forms: advection, divergence, rotational, and skew-symmetric. The conservation properties of these forms in terms of mass, momentum, and kinetic energy have been discussed in reference [23]. Among these four forms the skew-symmetric form satisfies the conservation property in all aspects and has been proved reliable in reference [24]. The mathematical expression for the skew-symmetric convection term in its tensor form is given by,

$$\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} \right)$$

where $i, j = 1, 2$ for the two-dimensional case. It is clear from Eq. (15), that the skew-symmetric form of the convection term is equal to the average of its divergence and the advection forms. The respective discretized system of equations using compact FD schemes for each of these terms in two dimensions is given below.

![Figure 1. Schematic of the staggered Cartesian grid.](image-url)
3.3.1.1. Divergence part of the convection term. This part contains the computation of the partial derivatives of the form $\frac{\partial q_u^2}{\partial x}$, $\frac{\partial q_v^2}{\partial y}$, $\frac{\partial q_u q_v}{\partial y}$, and $\frac{\partial q_v q_u}{\partial x}$. Among these four, the partial derivatives with respect to $x$ have to be obtained over horizontal lines, and the other over vertical lines. The stencil arrangement given in Figure 2a, considered along the pressure nodes with $\psi_0$ and $\psi_N$ representing values of the unknown at the physical boundary, is used to compute the derivatives of the square terms; whereas, partial derivatives of the cross product terms are computed over velocity nodes as given in Figure 2b, with $\psi_0$ and $\psi_N$ representing the boundary locations wherein the cross product of velocity has to be interpolated. The white circles in Figure 2a correspond to the pressure nodes and the filled circles in Figure 2b represent the vertices, wherein the velocity values have to be interpolated. For example, the compact system for $\frac{\partial q_u^2}{\partial x}$ with Dirichlet boundary conditions is given by

$$\frac{\partial u^2}{\partial x}_{\psi_1} - \frac{\partial u^2}{\partial x}_{\psi_2} = \frac{1}{\Delta x} \left(-u_{\psi_0}^2 + 2u_{\psi_1}^2 - u_{\psi_2}^2\right) + o(\Delta x^3) \quad (16)$$

$$\frac{\partial u^2}{\partial x}_{\psi_{i-1}} + 22 \frac{\partial u^2}{\partial x}_{\psi_i} + \frac{\partial u^2}{\partial x}_{\psi_{i+1}} = \frac{24}{\Delta x} \left(u_{\psi_{i+1}}^2 - u_{\psi_{i-1}}^2\right) + o(\Delta x^4) \quad i = 2, 3, \dots, N - 2 (17)$$

$$\frac{\partial u^2}{\partial x}_{\psi_{N-1}} - \frac{\partial u^2}{\partial x}_{\psi_{N-2}} = \frac{1}{\Delta x} \left(u_{\psi_N}^2 - 2u_{\psi_{N-1}}^2 + u_{\psi_{N-2}}^2\right) + o(\Delta x^3) \quad (18)$$

Thus, the system of equations will have the tridiagonal form of the coefficient matrix on the LHS. Similar set of equations, as given in Eqs. (16)–(18), have been derived for other terms.

3.3.1.2. Advection part of the convection term. This part contains the computation of the partial derivatives of the form $u \frac{\partial q_u^2}{\partial x}$, $\nu \frac{\partial q_v^2}{\partial y}$, $\nu \frac{\partial q_u q_v}{\partial y}$, and $u \frac{\partial q_v q_u}{\partial x}$. Among these four, once again the partial derivatives with respect to $x$ have to be obtained over horizontal lines and the remaining terms over vertical lines. The stencil

![Figure 2](image-url)
arrangement given in Figure 3a, considered along the pressure nodes, is used to compute \(\frac{u}{\psi_0}, \frac{v}{\psi_0}\) with \(\psi_0\) and \(\psi_N\) representing the values of the unknown at the physical boundary; whereas, \(\frac{v}{\psi_0}\) and \(\frac{u}{\psi_0}\) are computed over velocity nodes, as given in Figure 3b, with \(\psi_0\) and \(\psi_N\) representing the values of the unknown at the fictitious boundary point. It must be mentioned here that in the calculation of \(\frac{v}{\psi_0}\) and \(\frac{u}{\psi_0}\), the convective velocities \(v\) and \(u\) at their corresponding derivative locations are estimated using a four-point interpolation. The white circles in Figure 3a correspond to the pressure nodes, and the filled circles in Figure 3b represent the vertices of the computational cells. For example, the compact system for \(\frac{u}{\psi_i}\) with Dirichlet boundary conditions is as follows.

\[
\begin{align*}
\frac{\partial u}{\partial x} \bigg|_{\psi_i} + \frac{\partial u}{\partial x} \bigg|_{\psi_2} &= \frac{1}{2\Delta x} (-u_{\psi_0} - 4u_{\psi_1} + 5u_{\psi_2}) + o(\Delta x^3) \quad (19) \\
\frac{\partial u}{\partial x} \bigg|_{\psi_{i-1}} + \frac{\partial u}{\partial x} \bigg|_{\psi_i} &= \frac{1}{6\Delta x} (-u_{\psi_0} - 9u_{\psi_1} + 9u_{\psi_2} + u_{\psi_3}) + o(\Delta x^4) \quad (20)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} \bigg|_{\psi_{i-1}} + 4 \frac{\partial u}{\partial x} \bigg|_{\psi_i} + \frac{\partial u}{\partial x} \bigg|_{\psi_{i+1}} &= \frac{3}{4\Delta x} (u_{\psi_{i+1}} - u_{\psi_{i-1}}) + o(\Delta x^4) \quad i = 2, 3, \ldots, N - 2 \quad (21)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial x} \bigg|_{\psi_{i-1}} + \frac{\partial u}{\partial x} \bigg|_{\psi_2} &= \frac{1}{2\Delta x} (u_{\psi_N} + 4u_{\psi_{N-1}} - 5u_{\psi_{N-2}}) + o(\Delta x^3) \quad (22)
\end{align*}
\]

Equations (19) and (22) or (20) and (23) have to be used for the immediate neighbor of the boundary nodes. Thus, the system of equations will have the tridiagonal form of the coefficient matrix on the LHS. Similar set of equations, as given in Eqs. (19)-(23), have been derived for other terms.

3.3.2. Diffusion terms. Again, the schematic shown in Figure 3a illustrates the grid arrangement for the discretization of the diffusive terms \(\frac{\partial^2 u}{\partial x^2}\) or \(\frac{\partial^2 v}{\partial y^2}\), and
Figure 3b for $\frac{\partial^2 y}{\partial x^2}$ or $\frac{\partial^2 y}{\partial x^2}$. For example, the compact system for $\frac{\partial^2 u}{\partial x^2}$ with Dirichlet boundary conditions is given by

$$\frac{\partial^2 u}{\partial x^2}|_{\psi_0} + 12 \frac{\partial^2 u}{\partial x^2}|_{\psi_1} = \frac{1}{\Delta x^2} \left( \frac{167}{12} u_{\psi_0} - \frac{86}{3} u_{\psi_1} + \frac{31}{2} u_{\psi_2} - \frac{2}{3} u_{\psi_3} - \frac{1}{12} u_{\psi_4} \right) + o(\Delta x^3)$$

(24)

$$\frac{\partial^2 u}{\partial x^2}|_{\psi_{i-1}} + 10 \frac{\partial^2 u}{\partial x^2}|_{\psi_{i}} + \frac{\partial^2 u}{\partial x^2}|_{\psi_{i+1}} = \frac{6}{5\Delta x^2} \left( u_{\psi_{i+1}} - 2u_{\psi_{i}} + u_{\psi_{i-1}} \right) + o(\Delta x^4) i = 1, 2, \ldots, N - 1$$

(25)

$$\frac{\partial^2 u}{\partial x^2}|_{\psi_N} + 12 \frac{\partial^2 u}{\partial x^2}|_{\psi_{N-1}} = \frac{1}{\Delta x^2} \left( \frac{167}{12} u_{\psi_N} - \frac{86}{3} u_{\psi_{N-1}} + \frac{31}{2} u_{\psi_{N-2}} - \frac{2}{3} u_{\psi_{N-3}} - \frac{1}{12} u_{\psi_{N-4}} \right) + o(\Delta x^3)$$

(26)

Thus, the system of equations will have the tridiagonal form of the coefficient matrix on the LHS. Similar set of equations, as given in Eqs. (24)–(26) after taking care of the necessary grid orientations, are applicable to the remaining diffusive terms.

3.3.3. Pressure gradient term. For the discretization of the terms $\frac{\partial p}{\partial x}$ or $\frac{\partial p}{\partial y}$ which appear in the momentum equations, the schematic shown in Figure 2a has been used after replacing $u^2$ or $v^2$, wherever present, with pressure $p$. For example, in the compact system for $\frac{\partial p}{\partial x}$, we have used the same system of equations as given in Eqs. (16)–(18).

4. PRESSURE POISSON EQUATION

For the fractional step algorithm used in this work, the Poisson equation for the pressure correction ($\phi$) has to be deduced by taking the divergence of Eq. (8) to give

$$\nabla \cdot \nabla \phi = \frac{\nabla \cdot U^*}{\Delta t}$$

(27)

In our work, instead of discretizing Eq. (27) straightforward, we follow the convention of deriving the discrete equation for pressure correction through the continuity equation which involves the substitution of Eq. (8) into Eq. (4). Herein, we show the distinction between the conventional second order and compact fourth order discrete equations for pressure correction for the purpose of comparing the difference in the structure of the coefficient matrix of the unknown vector. The second order accurate discrete form of Eq. (4) at the $n + 1$ time level obtained at any cell center $(i, j)$ wherein the pressure node is located, is given by

$$\frac{u_{i+\frac{1}{2}, j} - u_{i-\frac{1}{2}, j}}{\Delta x} + \frac{v_{i, j+\frac{1}{2}} - v_{i, j-\frac{1}{2}}}{\Delta y} = 0$$

(28)
Using the discrete form of Eq. (8) in Eq. (28) gives,

\[
-\left(\frac{\phi_{i-1,j} + \phi_{i+1,j}}{\Delta x^2}\right) + 2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\phi_{ij} - \left(\frac{\phi_{i,j-1} + \phi_{i,j+1}}{\Delta y^2}\right) = \frac{1}{\Delta t}\left[\frac{u_{i-1,j} - u_{i+1,j}}{\Delta x} + \frac{v_{i,j-1} - v_{i,j+1}}{\Delta y}\right]
\]

(29)

Equation (29) involves the conventional five-point stencil arrangement. Similarly, the compact fourth order approximation of the first derivatives in Eq. (4) gives,

\[
\frac{\delta_x u_{ij}}{\Delta x(1 + \Delta x^2 \delta_x^2)} + \frac{\delta_y v_{ij}}{\Delta y(1 + \Delta y^2 \delta_y^2)} = 0
\]

(30)

where, for example, \(\delta_x u_{ij} = u_{i+\frac{1}{2},j} - u_{i-\frac{1}{2},j}\) and \(\delta_y^2 u_{ij} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\).

Using the discrete form of Eq. (8) into Eq. (30) yields,

\[
-\frac{\Delta t}{24} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\left(\phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}\right)
+ \frac{11\Delta t}{6} \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)\phi_{ij} - \frac{\Delta t}{12} \left(\frac{1}{\Delta x^2} - \frac{1}{\Delta y^2}\right)\left(\phi_{i,j-1} + \phi_{i,j+1}\right)
- \frac{\Delta t}{12} \left(\frac{11}{\Delta y^2} - \frac{1}{\Delta x^2}\right)\left(\phi_{i,j+1} + \phi_{i,j-1}\right) = \frac{11}{12\Delta x}\left(u_{i-\frac{1}{2},j} - u_{i+\frac{1}{2},j}\right)
+ \frac{11}{12\Delta y}\left(v_{i,j-\frac{1}{2}} - v_{i,j+\frac{1}{2}}\right) + \frac{11}{24\Delta x}\left(u_{i-\frac{1}{2},j+1} + u_{i-\frac{1}{2},j-1} - u_{i+\frac{1}{2},j+1} - u_{i+\frac{1}{2},j-1}\right)
+ \frac{11}{24\Delta y}\left(v_{i,j+\frac{1}{2}} + v_{i,j-\frac{1}{2}} - v_{i+1,j+\frac{1}{2}} - v_{i+1,j-\frac{1}{2}}\right)
\]

(31)

Equation (31) involves the nine-point stencil arrangement. Equations (29) or (31) have been solved after imposing \(\frac{\partial \phi}{\partial n} = 0\) on the boundaries using the BICGSTAB (2) algorithm. Further, in the pressure correction computation we have also adopted the inner iteration strategy as recommended by reference [29], which ensures the incompressibility constraint at every discrete time step.

5. NUMERICAL EXPERIMENTS

This section is devoted to assess the performance of the developed compact higher order numerical solver (NS). To start with, the spatial and temporal accuracy of the solver is tested by comparing the numerical and exact solutions of the decaying vortex problem governed by the NS equation. Later on, to ensure the robustness of the solver in simulating the fluid dynamic problems of academic and practical interest, two well explored benchmark problems, namely the steady lid driven cavity and backward facing step flows, have been simulated.
5.1. Decaying Vortices

The analytical solution to the velocities and pressure for the two-dimensional unsteady flow of decaying vortices in a domain bounded by \( x \in [0, 1] \) and \( y \in [0, 1] \) is given by

\[
\begin{align*}
    u(x, y, t) &= -e^{\left(\frac{-\pi^2}{Re}\right)} \sin(\pi x) \cos(\pi y) \\
    v(x, y, t) &= -e^{\left(\frac{-\pi^2}{Re}\right)} \cos(\pi x) \sin(\pi y) \\
    p(x, y, t) &= \frac{1}{4} e^{\left(\frac{-\pi^2}{Re}\right)} \left[ \cos(2\pi x) + \cos(2\pi y) \right]
\end{align*}
\]

To assess the temporal and spatial accuracy of the solver, a couple of numerical investigations have been conducted. The first investigation is the numerical simulation of the time evolution of decaying vortices for two different combinations of Reynolds numbers (Re) and nondimensional time (\( \tau \)), viz., \( Re = 1000 \) until \( \tau = 10.0 \) and \( Re = 10 \) until \( \tau = 0.3 \). In the evaluation of temporal accuracy, the \( L_2 \) error norm behavior of both velocities and pressure are compared; whereas, the \( L_2 \) error norm behavior of the pressure is emphatic enough to comment on the spatial accuracy of any incompressible flow simulations. Figures 4a and 4b illustrate the ability of the developed solver in producing the second order time accurate solutions for velocity and pressure. The fractional step method recommended by reference [31] is responsible for generating these high order solutions. These computations have been made with the time step ranging between \( 5 \times 10^{-4} \) and \( 5 \times 10^{-1} \) over a fixed grid spacing of \( \frac{1}{64} \).

Similarly, in Figures 5a and 5b, the pressure solutions obtained by refining the grid size gradually from \( 16 \times 16 \) to \( 256 \times 256 \) for a fixed time step of \( 5 \times 10^{-4} \), are plotted in order to evaluate the spatial accuracy of the developed solver. Here, the abscissa and the ordinate denote the number of nodes (N) and the \( L_2 \) error-norm,

Figure 4. \( L_2 \) Errors norms versus time step: (a) \( Re = 10 \) and \( \tau = 0.3 \); and (b) \( Re = 1000 \) and \( \tau = 10 \).
respectively. On comparing the error norm of the pressure patterns obtained for the same parametric study conducted in reference [24], it is apparent that our solver has attained the fourth order spatial accuracy.

5.2. Two-Dimensional Lid-Driven Cavity (LDC) Flow

This problem involves the solution of the NS equations in a unit square domain where the upper boundary moves with a uniform nondimensional velocity \( \mathbf{u} = (1, 0) \). No-slip boundary conditions are applied at all other boundaries \( \mathbf{u} = (0, 0) \). An earlier analysis on this problem reported in reference [34] shows that for \( \text{Re} \) greater than 8000, the first Hopf bifurcation occurs due to the steady solution losing its stability and attaining a steady periodic solution. Since our purpose is only to validate the developed solver, we restrict ourselves to the steady state solution regime. Hence, for this problem the numerical solutions have been simulated at two different Reynolds numbers, viz., \( \text{Re} = 400 \) and 7500. The \( u \) and \( v \) velocity profiles, respectively, about the geometric vertical centerline (g.v.c) and geometric horizontal centerline (g.h.c) of the cavity are the benchmark comparisons usually made on this problem. In our validation, at first for \( \text{Re} = 400 \), the grid independence study is performed using different grid spacings, viz., \( \frac{1}{32}, \frac{1}{48}, \frac{1}{64}, \frac{1}{96}, \frac{1}{108}, \) and \( \frac{1}{128} \). The actual and magnified images of the \( u \) velocity profiles about the g.v.c, for the abovementioned range of grids are compared in Figures 6a and 6b, respectively. The discrepancies among the results, in comparing them with Ghia et al. [35], are found to be negligible for the grid spacing beyond \( \frac{1}{128} \). Further verification to this assertion is achieved by showing that for \( \text{Re} = 400 \), the same \( 84 \times 84 \) grid is insufficient to produce comparable results with that of Ghia et al. [35] when the fourth order accurate spatial discretization is replaced with the second order counterpart. This feature is illustrated through the comparative plots of \( u \) and \( v \) velocity profiles shown in Figures 7a, 7b, 8a,
Figure 6. Comparison of versus component of velocity along g.v.c in a LDC at Re = 400, for various grid resolutions. (a) Actual profile and (b) magnified profile.

Figure 7. Comparison of versus component of velocity along g.v.c in a LDC at Re = 400, with a grid size of 84 × 84. (a) Actual profile and (b) magnified profile.

and 8b, respectively. Similarly, streamlines and vorticity fields for the same Re plotted in Figures 9a and 9b, respectively, also strengthens our contention. In order to check the reliability of the developed numerical method at high Re simulations, an extreme value of 7500, beyond which only steady periodic solution is reported, has been validated on a 160 × 160 grid. The streamlines and vorticity contours for this case, shown in Figures 10a and 10b, respectively, are found to have captured all essential patterns that are reported in an earlier work [35] for this Re, but with a grid size of 256 × 256.
5.3. Flow Over a Backward-Facing Step (BFS)

This problem deals with the incompressible laminar channel flow over a BFS of height \( h \). The length \( L \) and the height \( H \) of the downstream channel are chosen to be \( 40h \) and \( 2h \), respectively. At the inlet of the channel, a fully developed velocity profile represented by

\[
 u(0.5 \leq y \leq 1) = 24(1 - y)(y - 0.5)
\]

has been imposed. This profile yields the maximum \( u_{\max} \) and average inflow \( u_{\text{avg}} \) velocities of 1.5 and 1.0, respectively. The schematic of the computational domain with details to the

---

**Figure 8.** Comparison of versus component of velocity along g.h.c in a LDC at Re = 400, with a grid size of 84 x 84. (a) Actual profile and (b) magnified profile.

**Figure 9.** Streamlines and vorticity contours in a LDC at Re = 400, with a grid size of 84 x 84. (a) Streamlines and (b) vorticity.
geometry and boundary conditions is shown in Figure 11. Here, the Reynolds number is defined as $\text{Re} = \frac{u_{\text{avg}}}{\nu}$, where $\nu$ refers to kinematic viscosity. The prime objective to solve this problem is to verify the applicability of the present solver when applied to problems with outflow boundary conditions. The two representative Re of 200 and 800 are taken for this purpose and simulated on a computational domain with a grid spacing of $\frac{1}{40}$. The streamline patterns of the converged steady state solutions for both the cases are plotted in Figures 12a and 12b, respectively. The length of the recirculation zone ($x_1$) behind the step for both the Re, and the point of separation ($x_2$) and reattachment ($x_3$) of an upper eddy found at Re = 800, similar to reference [36], is given in Table 1. Further, the $u$ velocity profiles at $x = 7$ and $x = 15$ for Re = 800 shown in Figure 13a and 13b are found to be in good agreement with the results of Gartling [37].

Figure 10. Streamlines and vorticity contours in a LDC at $\text{Re} = 7500$, with a grid size of $160 \times 160$. (a) Streamlines and (b) vorticity.

Figure 11. Schematic of a BFS.
Figure 12. Streamline pattern for a flow over a BFS at (a) Re = 200 and (b) Re = 800.

Table 1. Comparison of locations of recirculation ($x_1$), separation ($x_2$), and reattachment ($x_3$)

<table>
<thead>
<tr>
<th>Reynolds number (Re)</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Re = 200</td>
<td>2.62</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Re = 800</td>
<td>6.002</td>
<td>4.85</td>
<td>10.3057</td>
</tr>
</tbody>
</table>

Figure 13. Comparison of the versus component of velocity profiles for a flow over a BFS at Re = 800 at two different sections along stream-wise direction of the flow. (a) $x = 7$ and (b) $x = 15$. 
6. CONCLUSION

In this study, a robust method of using higher order compact FD schemes to solve UINSE on a staggered grid has been analyzed. All of the necessary compact discretization procedures to various derivative terms in the governing equations have been clearly explained. An efficient factorization strategy, based on D'yakanov ADI-like scheme, adopted for the implicit part of the momentum equation has been highlighted. The developed scheme has the ease in implementation of Dirichlet or Neumann boundary conditions. Further, a novel approach of deriving fourth order accurate pressure Poisson solver on staggered grids has been introduced and compared with its second order counterpart. Finally, the validation of the developed solver with the benchmark solutions prove the improvement in the spatial accuracy attained while using compact finite difference schemes in place of conventional second order schemes.

REFERENCES