Flux scaling from near-wall coherent structures in turbulent convection.

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Abstract

We hypothesize that there are laminar natural convection boundary layers, forced by a shear due to the large-scale flow in the bulk, between the near-wall sheet plumes observed in turbulent Rayleigh - Bénard convection. In the absence of a mean wind, the classical $Ra^{1/3}$ scaling and the correct Pr dependence is obtained using the similarity solution for natural convection boundary layers and the stability condition of $Ra_\delta \sim 1000$. For weak shear, a fifth order algebraic equation, obtained from the order of magnitude analysis of integral boundary layer equations, is solved by perturbation methods to obtain the boundary layer thickness as a function of the Grashoff number (Gr) and the Reynolds number (Re). If the thermal dissipation from this boundary layer thickness is the major part of the total dissipation, using the stability condition of mixed convection boundary layers, we show that the effect of shear is to reduce the power law dependence of Nu on Ra.

1 Introduction

The question of interest is the scaling of non-dimensional flux, the Nusselt number (Nu) as a function of the driving parameter, Rayleigh number (Ra), and the Prandtl number(Pr) in turbulent convection. Here, $Nu = q/(\alpha\Delta T/H)$ is the ratio of the total flux to the diffusive flux, $Ra = g\beta\Delta T H^3/(\nu\alpha)$ is the ratio of the driving buoyancy forces to the restraining dissipative effects, $Pr = \nu/\alpha$, a fluid property, is the ratio of the rate of propagation of viscous to molecular diffusive effects, g = the acceleration due to gravity, β = the coefficient of thermal expansion, ΔT = the temperature difference between the walls, H = the fluid layer height, q = the kinematic heat flux, $Q/\rho C_p$ where, Q = the heat flux, ρ = the fluid density, C_p = the specific heat at constant pressure, ν = the kinematic viscosity and α = the thermal diffusivity.

At high Rayleigh numbers, the turbulent processes are important in the bulk and the dissipative effects near the walls. Dimensional reasoning implies that, $Nu \sim (RaPr)^{1/2}$ if bulk determines the heat flux while $Nu \sim Ra^{1/3}$, independent of layer height, if the boundary layers alone determine the flux. The experimentally observed anomalous scaling law is $Nu \sim Ra^n$ where n is slightly less than 0.3 [1]. Value of n close to $1/3$ implies that the flux is predominantly determined by the near wall phenomena, but not completely so, with the bulk having a weak role to play. The bulk was found to have a large scale flow (Figure 2(a)) by Krishnamurthi and Howard (see Siggia [15]), which causes a shear near the wall and could be one of the reasons for the anomalous flux scaling. Various theories, with different assumptions about the near wall boundary layers and their interaction with the wind exist to account for the anomalous flux scaling. Some of the suggested scenarios include a near wall mixing zone between the boundary layer and the bulk [2], turbulent shear boundary layer on the bottom and the top walls [14] and a Blasius boundary layer[6]. The nature of near wall boundary layers in turbulent Rayleigh - Bénard convection still remain a matter of controversy.

Contrary to these assumptions, experimental visualizations [17, 5, 9, 10] and numerical simulations [13] show sheet plumes near the top and bottom walls. Sheet plumes are buoyant fluid rising (falling) in the form of sheets from lines on the horizontal heated (cooled) surface. Figure 1 show the top view of near-wall

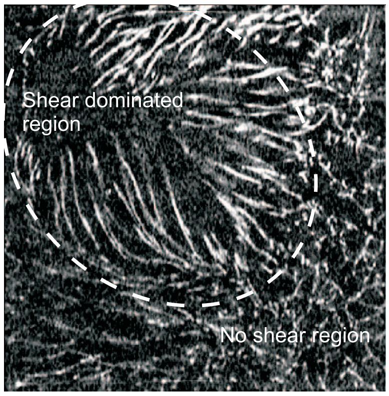


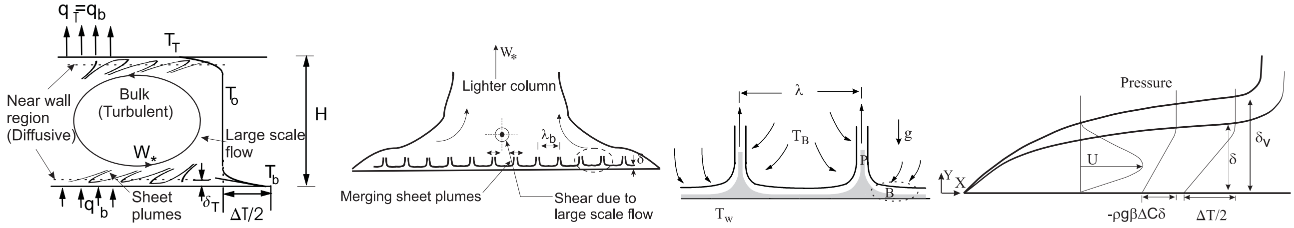
Figure 1: Top view of plume structure at $Ra \sim 10^{11}$[10]. The white lines are the top view of the merging sheet plumes

plume structure obtained by Puthenveetil[10] at $Ra = 2 \times 10^{11}$ and Schmidt number of 602. The lines in the figure are the top view of these rising sheets of lighter fluid. Theerthan & Arakeri [16] and Puthenveetil & Arakeri[9] hypothesized that these sheet plumes are outcomes of gravitational instability along lines on the laminar natural convection boundary layers [12] that form on the hot/cold surface. The sheet plumes move laterally and merge so that at any instant, the plumes form a complex pattern of lines on the hot surface when viewed from top(figure 1). The rising sheet plumes merge laterally due to the entrainment flow into these plumes, resulting in columns of rising fluid which drive a coherent large scale circulation in the convection cell(Figure 2(b)). This large scale flow in turn creates a shear near the wall and align the sheet plumes along the shear direction; figure 1 shows aligned sheet plumes over a major part of the bottom plate. We could expect this shear to modify the nature of the boundary layers between the sheet plumes so that the boundary layer thickness and the instability conditions are changed from that in Rotem & Classen[12]. The aim of the present paper is to show that it is possible to use a realistic approximation of the above near wall coherent structures in the Nusselt number integral relations to arrive at an anomalous flux scaling.

2 The no mean wind case

Puthenveetil and Arakeri [9] have shown that the sheet plume structure near the walls show complex structures with fractal nature, with spacings between them being log-normally distributed. Each sheet plume, on an average, is fed by laminar natural convection boundary layers on either side from a distance of BZ_w, where $Z_w = \nu\alpha/g\beta\Delta T_w$ is a near wall length scale in turbulent free convection and $B = 52$ for $Pr \sim 1$ and $B = 92$ for $Pr \sim 600$[16, 17, 9]. Alternatively, this implies that the plume structure can be approximated by a regular array of sheet plumes with two laminar natural convection boundary layers on either side[16, 17, 9] and having a mean spacing λ(Figure 2(c)) given by, $\lambda/Z_w = Ra_\lambda^{1/3} = B$, where, $Ra_\lambda = g\beta\Delta T_w\lambda^3/\nu\alpha$ is the Rayleigh number based on the mean plume spacing, ΔT_w is the near wall driving potential, equal to $\Delta T/2$ at high Ra.

Similarity solution for such boundary layers exist [12] which



(a) (b) (c) (d)

Figure 2: Schematic of turbulent Rayleigh Bénard convection (a)Front view, (b) Side view of Figure 2(a) showing the sheet plumes giving rise to the plume column that drive the large scale flow, (c)Zoomed view of the dashed ellipse in Figure 2(b), (d) Zoomed view of the dashed ellipse in Figure 2(c).

show that the thermal boundary layer thickness,

$$\frac{\delta(x)}{\lambda} = \eta_\delta \left(\frac{x}{\lambda}\right)^{2/5} Gr_\lambda^{-1/5} \tag{1}$$

where $Gr_\lambda = Ra_\lambda/Pr$ the Grashoff number based on λ, x is the horizontal distance along the boundary layer and $\eta_\delta = 5Pr^{-1/3}$ is the value of the similarity variable $\eta = (y/\lambda)(x/\lambda)^{-2/5}Gr_\lambda^{1/5}$ at $y = \delta(x)$ with y being the vertical co-ordinate direction. The maximum or the critical value of the boundary layer thickness, at $x = \lambda/2$, is

$$\delta_c = \delta|_{x=\lambda/2} = 5 \times 2^{-2/5} \lambda Ra_\lambda^{-1/5} Pr^{-8/15} \tag{2}$$

Equation (2) can also be written in dimensionless form as a relationship between $Ra_\delta = g\beta\Delta T_w \delta_c^3/\nu\alpha$, the Rayleigh number based on the critical boundary layer thickness and Ra_λ as

$$Ra_\lambda = 8 \times 5^{-15/2} Ra_\delta^{5/2} Pr \tag{3}$$

By taking the spatial average of $k\left.\frac{\partial T}{\partial y}\right|_{y=0} / k\frac{\Delta T}{H}$ over λ, where k is the thermal conductivity of the fluid, using the similarity variables of [12], Theerthan and Arakeri[16] showed that

$$Nu = \frac{-5 \times 2^{1/15}}{6} H'(0) Ra_\lambda^{-2/15} Pr^{-1/5} Ra^{1/3} \tag{4}$$

where $-H'(0) = 0.38Pr^{0.27}$ is the value of first derivative of the similarity temperature function $H(\eta) = (T(y) - T_B)/\Delta T_w$ at $\eta = 0$ and T_B is the temperature of the fluid away from the walls. Using (3), (4), and the stability condition that,

$$Ra_\delta \sim 1000, \tag{5}$$

we get,

$$Nu = 0.126 Ra^{1/3} Pr^{-0.06}, \tag{6}$$

the classical flux scaling relation with a weak Prandtl number dependence similar to the experimentally observed $Pr^{-0.03}$ dependence[19].

3 The effect of a mean wind

Figure 1 shows that at $Ra > 10^7$, when the large scale flow is prevalent, the planform show regions with and without the effect of shear. The scaling of (6) could be expected to hold in the no shear region. In the shear dominated regions the laminar natural convection boundary layers feeding the plume would be modified due to the presence of the external shear (Figure 3(c)). The boundary layers then become mixed convective in nature which have non-negligible buoyancy and shear effects in them[11]. To estimate the Nu, no similarity solution for such boundary layers seems to be available to directly calculate the vertical temperature gradient at the wall, as was done for (4). We hence use the integral relation for scalar variance for steady turbulent Rayleigh - Bénard convection, along with appropriate estimate of thermal dissipation to find the flux scaling. The analysis is limited to estimating only the Ra dependence; the Pr effects could be included in future.

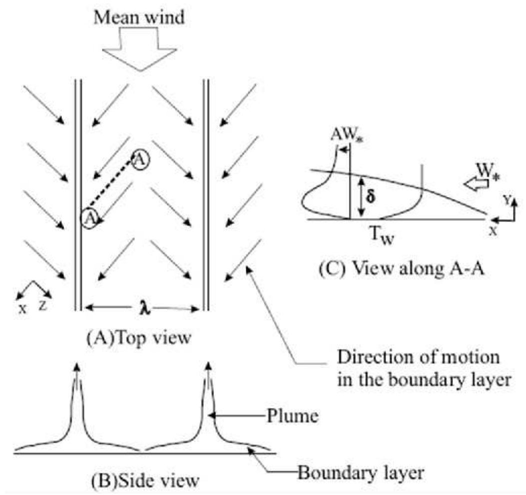


Figure 3: (a)Top view of the aligned plumes due to a mean wind. (b) A front view in a vertical section perpendicular to the aligned sheet plumes. (c) The boundary layer with external shear at the vertical section A-A.

The integral scalar variance equation

$$\langle\epsilon_\theta\rangle_V = \alpha\frac{(\Delta T)^2}{H^2} Nu \tag{7}$$

give Nusselt number in terms of volume averaged thermal dissipation, $\langle\epsilon_\theta\rangle_V$[3]. Using appropriate estimate for the thermal dissipation in terms of Ra, one can find a scaling relation between Nu and Ra. As a first approximation, we assume that the volume averaged dissipation is dominated by the dissipation inside the boundary layers. This is justifiable because diffusive transport is predominant inside the boundary layers while turbulent transport dominate the bulk. Various experiments[4, 7] and numerical simulations[18] of thermal dissipation inside the boundary layers corroborate this assumption. Now, the flux scaling predicted by the theory will mostly depend on the correctness of the approximation of the near-wall structure.

Assuming a regular array of sheet plumes fed by mixed convection boundary layers on either side as the coherent structure on the hot and cold walls, the volume averaged thermal dissipation, if only the near wall boundary layers and plumes contribute to the total thermal dissipation, can be estimated as

$$\langle \epsilon_\theta \rangle_V = \frac{\int_0^\lambda \int_0^{\delta(x)} \alpha \left(\frac{\partial T}{\partial y}\right)^2 dydx}{\int_0^\lambda \int_0^{\delta(x)} dydx} \frac{\delta_c}{H} \sim \frac{\alpha \Delta T^2}{\delta_c H}. \tag{8}$$

Since the contribution to thermal dissipation from the near wall sheet plumes will also be of the same functional form and magnitude as the boundary layer contribution given by equation(8), the contribution of sheet plumes will only change the prefactor in (8). (See appendix B of Theerthan and Arakeri [16]). Equating (8) to (7) we get,

$$Nu \sim \frac{H}{\delta_c}. \tag{9}$$

However, $\delta(x)$ is now different from (1) due to the external shear. Same is the case for the critical boundary layer thickness since the stability condition $Ra_\delta \sim 1000$ will also change due to the external shear.

We estimate δ_c using an order of magnitude analysis of the integral equations for mixed convection boundary layers. The laminar natural convection boundary layer equations are integrated using the following boundary conditions

$$\text{at } y = 0 : u = 0, v = 0, T = T_w, \tag{10}$$

$$\text{at } y = \delta : u = AU_{nw}, T = T_B, \tag{11}$$

where, $U_{nw} = c_1 W_*$ is the near wall shear velocity proportional to the Deardorf bulk velocity scale $W_* = (g\beta qH)^{1/3}$, A is another constant to take into account the angle at which the shear will act with respect to the direction of W_* (see Figure 3) and c_1 is a pre-factor. The integral forms of the continuity, x-momentum, y-momentum and the energy equations are respectively,

$$\frac{d}{dx}\int_0^\delta u dy - u\frac{d\delta}{dx} + v_\delta = 0, \tag{12}$$

$$\frac{d}{dx}\int_0^\delta u^2 dy - (AU_{nw})^2 \frac{d\delta}{dx} \tag{13}$$

$$+AU_{nw}v_\delta + \frac{1}{\rho}\int_0^\delta \frac{\partial p}{\partial x} dy + \nu \left(\frac{\partial u}{\partial y}\right)_{y=0} = 0,$$

$$p + \rho g \beta \int_y^\delta (T - T_\infty)\, dy = 0, \text{ and} \tag{14}$$

$$\frac{d}{dx}\int_0^\delta u (T - T_\infty)\, dy + v_\delta T_\infty = -\alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{15}$$

Using the following scales $u \sim U_c$, $y \sim \delta_c$, $x \sim \frac{\lambda}{2}$, where U_c is the characteristic velocity inside the boundary layer, the above equations result in an order of magnitude balance equation

$$Gr_\lambda \left(\frac{\delta_c}{\lambda/2}\right)^5 + Re_\lambda \left(\frac{\delta_c}{\lambda/2}\right)^2 - 2 \sim 0. \tag{16}$$

for the critical boundary layer thickness[8]. Here, $Re_\lambda = (AU_{nw}\lambda/2)/\nu$ is the Reynolds number based on the plume spacing, which can be written as,

$$Re_\lambda = \frac{Ac_1}{2} Re_H \left(\frac{Ra_\lambda}{Ra}\right)^{1/3}, \text{where} \tag{17}$$

$$Re_H = \frac{W_* H}{\nu} = \left(\frac{RaNu}{Pr^2}\right)^{1/3}. \tag{18}$$

The expression (16) for the critical boundary layer thickness in the presence of an external shear has the correct asymptotes. When $Re_\lambda \to 0$, $\delta_c \to \lambda/Gr_\lambda^{1/5}$ given by (2), while when $Gr_\lambda \to 0$ $\delta_c \to \lambda/\sqrt{Re_\lambda}$ the Blasius boundary layer solution. Equation (16) can solved by perturbation methods with Re_λ/Gr_λ as the small parameter to find an expression for the critical boundary layer thickness for small Re_λ/Gr_λ as[8],

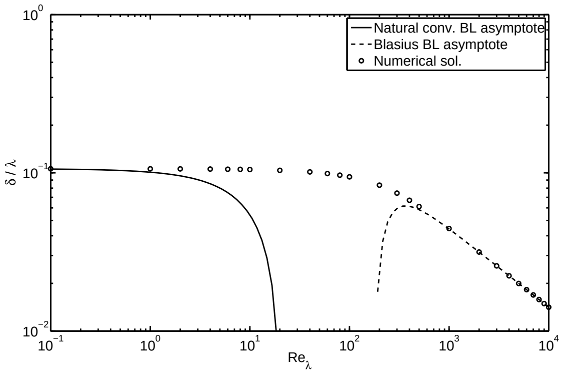


Figure 4: Comparison of the perturbation solution (19) with the numerical solution of (16) along with the Blasius boundary layer asymptote.

$$\frac{\delta_c}{\lambda} \sim \left(\frac{2}{Gr_\lambda}\right)^{1/5} - \frac{Re}{2^{2/5}5}\left(\frac{1}{Gr_\lambda}\right)^{3/5} - \frac{1}{25}\left(\frac{Re_\lambda^2}{Gr_\lambda}\right) + \ldots \tag{19}$$

Equation (19), when rewritten in terms of Ra_λ and Ra_δ retains the same form as (3), except with a change in the prefactor. Neglecting the Pr dependence for the present, equations (9), (17), (18),(3) and the first two terms on the right of (19) imply

$$\frac{Nu}{Ra^{1/3}} \sim \frac{D}{Ra_\delta^{1/6}\left(c_2 Ra_\delta^{1/6} - c_3 Nu^{1/3}\right)}, \tag{20}$$

where c_2 and c_3 are constant pre-factors. Since $Nu/Ra^{1/3}$ is a constant for the case of no shear (6), if the stability condition remains same as in the no shear case, then (20) implies that $Nu/Ra^{1/3}$ increases with increase in Ra or $Nu \sim Ra^n$, where $n > 1/3$. This is understandable as the boundary layer thickness with external shear will be less than that with shear by (19) and Nu is inversely proportional to δ_c by (9). We now have to consider the change in stability condition (5) due to the external shear. Castaing et al.[2] show that,

$$Ra_\delta = F(Pr) + c_4 Pr Re_\delta^2, \tag{21}$$

where F is a weak function of Pr and c_4 is a constant pre-factor. Using $Re_\delta = Re_\lambda \delta_c/\lambda$ and the expression $Re_\delta = c_3\left(Ra_\delta^{5/2} Nu\right)^{1/3}$, where c_3 is a constant pre-factor, obtained from (17) and (3) along with (21) implies that variation of $\chi = Ra_\delta^{1/3}$ in the mean wind case is given by a cubic equation

$$\chi^3 - C_3\chi^2 - C_4 = 0 \tag{22}$$

where,

$$C_3 = K\left(I - J\left(Nu/\sqrt{Ra}\right)^{1/3}\right)^2 Nu^2 \tag{23}$$

and C_4 is a constant. The solution of (22) gives χ as a polynomial in C_3, which can however be approximated as

$$Ra_\delta^{1/3} \sim C_3, \tag{24}$$

where C_3 is given by (23). By substituting (24) in (20) and expanding the right hand side in terms of Nu gives

$$\frac{Nu}{Ra^{1/3}} \sim \frac{1}{Nu^{4/3}} + \frac{1}{Ra^{1/6}Nu} + \ldots \tag{25}$$

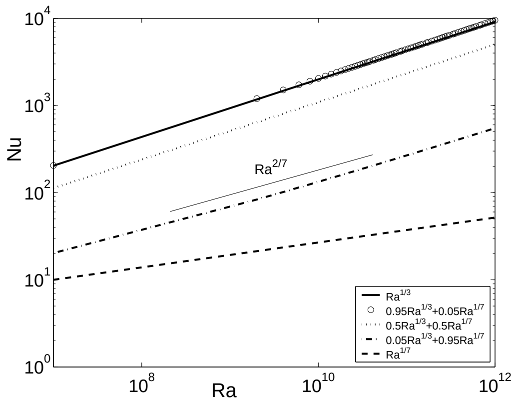


Figure 5: The flux scalings obtained by various area fractions affected by shear.

Considering only the first term on the right hand side of (25) we get the flux scaling as

$$Nu \sim Ra^{1/7}, \tag{26}$$

a much lower exponent than the classical exponent of 1/3 or the experimentally observed exponent of approximately 0.3. However it needs to be noticed that at any Ra, only a part of the area of the bottom and top plate will have substantial shear effects, as can be seen from Figure 1. So flux will scale as (26) over the area where shear effects are predominant and as (6) where shear effects are negligible. To see the effect of this mixed scaling of flux over the area of the plates we plot the summation of different combinations, ranging from shear effects over 5% of area to 95% of areas, in Figure 5. It is seen that the effect of shear on flux scaling is weak, shear over 95% area reduces the exponent to about 2/7. This same weak dependence of large scale flow on the flux scaling could also be inferred from Figure 4, where the boundary layer thickness remains same as that without shear till a Re_λ of about 100, corresponding to a Rayleigh number of approximately 10^9.

4 Conclusions and Discussion

We have shown that assuming the near-wall coherent structures in turbulent convection to be sheet plumes, resulting from the instability of laminar natural convection boundary layers on either side, the classical $Nu \sim Ra^{1/3}$ scaling is obtained if the stability condition $Ra_\delta \sim 1000$ holds (eq.(6)). In the presence of shear due to the large scale flow, using perturbation solution to the order of magnitude equation (16) obtained from the integral equations (12) to (15), we obtained the expression for the boundary layer thickness in the presence of the mean wind (eq.19). An external shear always reduces the boundary layer thickness which results in an increase of scaling exponent of $Nu - Ra$ relationship if the boundary layer instability condition is same as that in the case of no shear (20). Considering the stability condition given by Castaing[2] we show that the presence of shear on the near wall coherent structures in turbulent convection is to reduce the flux scaling. The exact exponent in the $Nu - Ra$ relationship will depend on the area of the plates affected by the shear. The above analysis is approximate in many accounts and needs to be taken only as a direction of future research in understanding the flux scaling in turbulent convection. More rigorous analysis including the Pr effects and a rigorous accounting of the pre-factors for the mean wind case have to be done. One could also use the available solutions for mixed convection boundary layer[11] to explore the flux scaling.

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