Analytical solutions for convective fragmentation

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1. Introduction

Fragmentation (or comminution) of particles occurs in many processes involving convection of suspensions in liquid or gas carrier phases. Such fragmentation is often spatially nonhomogeneous (due, for example, to turbulence induced fragmentation). Recently Nere and Ramkrishna (2005, 2006), Ramkrishna and Nere (2006), Kostoglou and Karabelas (2004, 2007), and Kostoglou (2006), Kostoglou and Ramkrishna (2004, 2007), and Kostoglou (2006) have studied a class of pipe flow problems involving convective fragmentation. In the present paper we generalize a version of the problem discussed above to allow for unsteady two-dimensional axisymmetric convective fragmentation and report a corresponding analytical solution. We connect our analytical solution to the grinding solution of Reid (1965) and demonstrate its applicability by applying it to a problem of a pulsating convective fragmentation in an unsteady axisymmetric jet, with the fragmentation characteristics being spatially dependent.

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2. Governing equations

Consider a two-dimensional axisymmetric multiphase fragmenting flow described by the respective radial and axial cylindrical polar co-ordinates r and z, the latter being coincident with the axis of symmetry. The system consists of one continuous incompressible carrier fluid phase and M particulate phases, each being associated with a discrete diameter and all being treated as interpenetrating continua, as in the classical Eulerian description. As mentioned above, the total discrete phase volume fraction is assumed to be small and all the discrete phases are assumed to be in velocity equilibrium with the carrier phase which has axial and radial velocity components u(r,z) and w(r,z). We will employ the convention that i=1 corresponds to the largest particle size. Fragmentation is handled by allowing the particulate phases to exchange mass. This exchange is modeled by defining breakage frequencies \( \Gamma_i(r,z) \) for each particulate phase which describe the frequency of breakup of drops of phase i and distribution coefficients \( n_i(r,z) \) which give the number of daughter drops of phase j per fragmenting drop of phase i. The breakage frequencies and distribution coefficients are made to depend on the independent variables to allow for the non-homogeneous fragmentation mentioned above. Neglecting diffusion, a mass balance transport equation can be written for the number density \( N_i(r,z,t) \) of particulate phase i of the form

\[
\frac{\partial N_i}{\partial t} + w \frac{\partial N_i}{\partial r} + u \frac{\partial N_i}{\partial z} = -\Gamma_i N_i + \sum_{k=1}^{i-1} n_k \Gamma_k N_k
\]  

(1)

The initial and boundary conditions are given by
\[ N_i(r, z, t) = N_{i,0}(r, t); \quad \frac{\partial N_i(0, z, t)}{\partial r} = 0 \] (2)
The exchange coefficients must satisfy
\[ \sum_{i=k+1}^{M} n_{ik} v_i = v_k \] (3)
which expresses conservation of mass during a single breakage event. Here \( v_i \) is the volume associated with the \( i \)th bin. Eq. (1) illustrates that fragmentation equations can always be solved in a sequential manner beginning with the equation for the largest drop size and proceeding successively to smaller drop sizes.

3. Adaptation of Reid’s solution to fragmentation in plug flow

Let us first consider the case where the mixture is being steadily transported by a constant axial velocity in a duct of length \( L \). Therefore \( N_i = N_i(z); w = 0 \) and \( n_i \) and \( \Gamma_i \) are constants. For this situation, Eqs. (1) and (2) simplify to
\[ u \frac{dN_i}{dz} = \Gamma_i N_i + \sum_{k=1}^{i-1} n_{ik} \Gamma_k N_k \] (4)
\[ N_i(0) = N_{i,0} \] (5)
These equations are analogous to those describing batch grinding processes, with the quantity \( z/u \) in Eq. (4) being analogous to time in the batch grinding problem. Eq. (4) admits an analytical solution which was first developed by Reid (1965). The corresponding problem of a plug flow aggregator was discussed by Nere and Ramkrishna (2005).

Eq. (4) can be solved analytically as reported by Bhamidipati et al. (2006). The solution is a slight extension of that presented by Reid (1965) to the case of non-zero initial conditions for all phases. The solution can be written
\[ N_i(z) = \sum_{k=1}^{i} A_{ik} \exp \left( -\frac{\Gamma_k z}{U} \right) \] (6)
where the \( A_i \)'s are related by the recurrence relationships.
\[ A_{ik} = \begin{cases} \frac{n_i \Gamma_i A_{ik}}{\Gamma_i - \Gamma_k} & \text{for } k = 1, 2, \ldots i - 1 \\ N_{i,0} - \sum_{l=1}^{i-1} A_{ik} & \text{for } k = i \end{cases} \] (7)
The combination of Eqs. (6) and (7) constitutes a closed form solution to Eq. (4). For future reference, it is useful to recognize that Eq. (6) can also be written in the form
\[ N_i(z) = \sum_{k=1}^{i} N_{k,0} \tilde{F}_{ik}(z) \] (8)
In other words, \( N_i \) is a linear combination of \( N_{1,0}, N_{2,0}, \ldots, N_{i,0} \). It is possible to generalize the solution described by Eqs. (6) and (7) in several ways. One example is presented below.

4. Generalization of Reid’s solution to fragmentation in general axisymmetric flow

Consider the general problem defined by Eq. (1). This problem involves an arbitrary steady axisymmetric carrier fluid flow (not necessarily restricted to flow in a duct) with unsteady fragmentation occurring in the discrete phases. A form of von Mises transformation (equivalent to a type of characteristics analysis) can be used to transform Eq. (1) from the independent variables \( r, z \) and \( t \) to the independent variables \( r_0, z \) and \( t \) where the relationship
\[ r = r(z, r_0) \] (9)
is determined by the solution to the problem
\[ \frac{dr}{dz} = \frac{w}{u} \] and \( r(0, r_0) = r_0 \) (10)
This produces
\[ \frac{\partial N_i}{\partial t} + u \frac{\partial N_i}{\partial z} = -\Gamma_i N_i + \sum_{k=1}^{i-1} n_{ij} \tilde{F}_{ik} N_k \] (11)
and
\[ N_i(r_0, 0, z, t) = N_{i,0}(r_0, t) \] (12)
with \( N_i \) now a function of \( r_0, z \) and \( t \). In a time independent situation, Eq. (11) is in effect a system of ordinary differential equations for a given \( r_0 \) which can be solved numerically by standard techniques. This is the discrete analog to the method of characteristics solution to the continuous population balance equations discussed in the book by Ramkrishna (2000). In the case of linear mass transfer terms, the steady form of Eq. (11) is analogous to Eq. (4) and the solution for each \( r_0 \) will be given by Eqs. (6) and (7). An example of application of similar transformations to the case of pure agglomeration is provided by Nere and Ramkrishna (2005).

Now consider the special case for which
\[ \Gamma_i(r_0, z) = \tilde{\Gamma}_i(r_0, z) \] (13)
(all breakage frequencies proportional) and the mass transfer terms are linear. For this special case, we can define a new independent variable (in place of \( z \)) following Austin and Bagga (1981) as
\[ \xi = \int_{r_0}^{z} \frac{\Gamma(r_0, z)}{u(r_0, z)} \, dz \] (14)
and transform Eqs. (11) from independent variables \((r_0, z, t)\) to independent variables \((r_0, \xi, t)\) to get
\[ \frac{\partial N_i}{\partial t} + u \frac{\partial N_i}{\partial \xi} = -\tilde{\Gamma}_i N_i + \sum_{k=1}^{i-1} n_{ij} \tilde{F}_{ik} N_k \] (15)
subject to
\[ N_i(r_0, 0, t) = N_{i,0}(r_0, t) \] (16)
Let \( \tilde{h}(s) \) be the Laplace transform of \( h(t) \). Taking the Laplace transform of Eq. (15) yields
\[ s\tilde{N}_i + \tilde{\frac{\partial N_i}{\partial \xi}} = -\tilde{\Gamma}_i \tilde{N}_i + \sum_{k=1}^{i-1} n_{ik} \tilde{F}_{ik} \tilde{N}_k \] (17)
By defining
\[ \tilde{M}_i(r_0, \xi, s) = \tilde{N}_i(r_0, \xi, s) \exp(s\zeta) \] (18)
Eq. (17) can be rewritten as
\[ \tilde{\frac{\partial M_i}{\partial s}} = -\tilde{\Gamma}_i \tilde{M}_i + \sum_{k=1}^{i-1} n_{ik} \tilde{F}_{ik} \tilde{M}_k \] (19)
As above, these equations can be solved for a given \( r_0 \) to determine \( \dot{M}_i \) and then \( \dot{N}_i(r_0, \xi, s) \). The solution to Eq. (19) is of the form of Eq. (8). The solution for \( \dot{N}_i(r_0, \xi, s) \) can therefore be written as

\[
\dot{N}_i(r_0, \xi, s) = \sum_{j=1}^{i} \dot{N}_{j,0}(r_0, s) \exp(-s\xi) F_g(r_0, \xi)
\]

where the inverse Laplace transform is of the form

\[
\dot{N}_i(r_0, \xi, t) = \sum_{j=1}^{i} \dot{N}_{j,0}(r_0, t - \xi) H(t - \xi) F_g(r_0, \xi)
\]

with \( H(x) \) being the Heaviside function. This solution can be written in the more convenient form

\[
\dot{N}_i(r_0, \xi, t) = \sum_{k=1}^{i} A_{ik} \exp(-\dot{R}_k \xi)
\]

where the \( A \)'s are related by the recurrence relationships

\[
A_{ik} = \left\{ \begin{array}{ll}
\sum_{j=k}^{i-1} n_{ij} \dot{R}_j A_{jk} & \text{for } k = 1, 2 \ldots i - 1 \\
\dot{N}_{i,0}(r_0, t - \xi) H(t - \xi) - \sum_{j=1}^{i-1} A_{ij} & \text{for } k = i
\end{array} \right.
\]

Eqs. (22) and (23) present the general solution for the unsteady fragmentation process occurring in a general steady two-dimensional axisymmetric flow field of the carrier phase. This in itself is novel and is applicable to several situations. The special case of pulsating drop injection will be explored in some detail in the subsequent section to illustrate the general solution’s utility.

5. Results and discussion

We herein present results based on the analytical solutions in Sections 3 and 4. The number of drop phases, \( M \) is varied while requiring the phase volumes to follow a geometric progression between the respective maximum and minimum values of volumes \( V_1 \) and \( V_M \) with the geometric ratio being \( g \). The distribution coefficients were calculated from

\[
n_{ij} = \left\{ \begin{array}{ll}
g^{j-i-2} - g^{i-1} & \text{for } i = j + 1, \ldots M - 1 \\
g^{M-j-2} - g^{M-j-1} & \text{for } i = M
\end{array} \right.
\]

Eq. (24) satisfies Kolmogorov’s hypothesis that the number of daughter drops of phase \( i \) per fragmenting drop of phase \( j \) is independent of the size of phase \( i \). The breakage frequencies were calculated from

\[
\dot{R}_i = 10^{g^2 - 1 - g^{M-1}}
\]

Eq. (25) is similar to one suggested by Kumar and Ramakrishna (1996). While Eqs. (24) and (25) are reasonable, they were chosen herein purely for the purposes of illustration and any other forms of distribution coefficients and breakage frequencies could have equally well been used.

First, consider fragmenting plug flow. Fig. 1 is a plot of Reid’s analytical solution presented in Eqs. (6) and (7). Here, we have chosen the representative values \( u = 0.01 \text{ ms}^{-1}, L = 2 \text{ m}, v_1 = (\pi/6) \times 250^3 \text{ km}^3 \text{ and } v_M = (\pi/6) \times 1^3 \text{ km}^3 \text{ (not intended to characterize any real fragmentation process exactly) and assumed that all drops at the entrance were of the largest size class. The results in this figure indicate classical fragmentation behaviour where the phase associated with the largest size exhibits an exponential decay, while the intermediate size phases initially increase and decrease.}

Second, consider fragmenting flow in a steady axisymmetric jet with pulsating drop injection. The solution given by Eqs. (22) and (23) can be adapted to this case, with the velocity components given by Schlichting (2004) as

\[
u(r, z) = \frac{32\theta r^2 (z + z_0)^3}{(\gamma r^2 + 4(z + z_0)^2)^2}
\]

\[
w(r, z) = \frac{4\theta r^2 (z - r^2 r^2 + 4(z + z_0)^2)}{(\gamma r^2 + 4(z + z_0)^2)^2}
\]

Here, \( \theta \) is the kinematic viscosity of the carrier fluid, \( \gamma \) is a constant (a measure of the jet momentum flux) and \( z_0 \) (chosen here to be equal to 0.01 m) is the virtual injector position which is introduced in order to eliminate the singularity associated with a point source. According to algebraic turbulence models, for example those discussed in Schlichting (2004), the effective turbulent kinematic viscosity is constant in a fully developed turbulent jet. Therefore, this flow field can be used to describe both laminar and turbulent jet flows. Substituting Eqs. (26) and (27) into Eq. (10) yields the carrier phase streamline equation:

\[
r(r_0, z) = 2r_0(z + z_0) \frac{z_0}{z^2 r_0^2 + 4z^2 + 4z_0^2}
\]

Using Eq. (28) to eliminate \( r \) in favor of \( r_0 \) in Eq. (26) leads to

\[
u(r_0, z) = \frac{2\theta r_0^2 (z - r_0^2 r_0^2 + 4z_0^2)}{(z + z_0)^2 (1/2 r_0^2 + 4z_0^2)}
\]

For the purposes of demonstrating the ability of the analytical solution to handle non-homogeneous fragmentation (such as that caused by turbulence, for example) we now choose the idealized breakage frequency expression

\[
\Gamma(r_0, z) = \exp \left(-\lambda \frac{z_0}{r_0^2} \right) * \exp \left(-k \frac{z_0^2}{r_0^2} \right)
\]

which allows the frequency to decay with both axial and radial distance from the nozzle; with \( \lambda \) and \( k \) being the corresponding decay parameters, \( L \) being an axial length scale, and \( R_0 \) being a measure of the injector radius. In addition, we characterized \( n_{ij} \) and \( \dot{R}_i \).
Fig. 2. Plot of number density $N_1$ (inset $N_5$) versus axial position on the centerline at $t=6s$

using Eqs. (24) and (25). We choose $M=20$, $v_1=1(\pi/6)3043 \mu m^3$, $v_1=(\pi/6)13 \mu m^3$, $\lambda=5$, $\gamma=32$, $R_0=6.5 \times 10^{-4}$ and $\theta=10^{-5} m^2/s$.

Two injection conditions are chosen. Firstly,

$$N_{i,0}(r_0,t) = \begin{cases} 
1 - \cos(\Omega t) & \text{for } i = 1 \\
0 & \text{otherwise}
\end{cases} \quad (31)$$

where $\Omega$ is the frequency of injection (chosen here to be equal to $10 \pi s^{-1}$) and with the inlet conditions normalized to where one drop enters the pipe at the inlet. The second injection condition is the same as above, except that only a single pulse is injected. The evolution of the drops of different size classes resulting from the fragmentation process was calculated using the solution presented in Eqs. (22) and (23). The result of the integration in Eq. (14) can be expressed in closed form in terms of incomplete exponential integrals for the forms of $u$ and $\tilde{u}$ depicted in Eqs. (29) and (30). The general result is lengthy and has been omitted for the sake of brevity. As an example, for the special case of $\tilde{u} = 1$ ($\lambda = k = 0$)

Fig. 3. Plot of SMD versus $r$ and $z$ at $t=6s$

Fig. 4. Plot of volume fraction versus $r$ and $z$ for $\kappa = 0$ and $5$; $t=1.8s$ and $2.1s$.

However, the periodic injection pattern described by Eq. (31) is still increases with increasing \( z \). \( N_c \) and \( N_r \) are obtained. Here \( c_0 \) is the jet momentum flux. The results presented below are for the more general case of \( \lambda \neq 0 \) and \( \kappa \neq 0 \).

Fig. 2 shows two axial number density distributions at \( t = 6 s \) for \( \kappa = 0.5 \). As can be observed from the figures, \( N_c \) decreases and \( N_r \) increases with increasing \( z \), as expected from the previous results. However, the periodic injection pattern described by Eq. (31) is still retained.

Fig. 3 depicts the axial and radial variations of the Sauter Mean Diameter

\[
\text{SMD} = \left( \frac{6}{\pi} \right)^{1/3} \frac{\sum_{i=1}^{M} N_i \nu_i}{\sum_{i=1}^{M} N_i \nu_i^{2/3}}
\]  

As expected the SMD decreases with increasing axial position and reaches an asymptotic value due to the exponential form of the breakage frequency given in Eq. (30). In addition, the inclusion of radial dependence of the breakage frequency in Eq. (30) leads to a higher SMD on the outer periphery and a smaller SMD in the core region of the jet.

Fig. 4 is a plot of the ratio of the total volume fraction to its maximum value at the nozzle,

\[
\alpha = \frac{1}{\nu_1} \sum_{i=1}^{M} N_i \nu_i
\]

for the case of a single pulse injection for two values of \( \kappa \) and at two instants of time. This plot illustrates the fact that the shape of the injection pulse changes as it propagates downstream as well as with variation in \( \kappa \).

### 6. Conclusion

In the foregoing, an analytical solution to the generalized problem of axisymmetric flow accompanied by unsteady fragmentation was presented. This solution can be thought of as a direct extension of Reid’s grinding solution. The solution was applied both to steady fragmentation in plug flow and to unsteady fragmentation with axially and radially decaying fragmentation frequencies in axisymmetric jet flow. Representative results were presented graphically.

The results discussed in this paper illustrate the capabilities of the general solution presented in Section 4. In the examples discussed explicitly above, all of the analysis could be carried out in closed form. While the generalization of Reid’s solution developed in Section 4 can always be expressed in closed form, the corresponding dependent variable transformations would have to be carried out numerically in many cases. It is believed that the numerical work required to do this will be considerably less than that associated with a full CFD approach. It is hoped, therefore, that the results presented in this paper will be of use to investigators seeking to verify aspects of CFD analysis in situations involving convective fragmentation.

### References


